清華大學

随机过程作业题

二阶矩过程部分

1. 解:

$$E[X(t)] = \frac{1}{T}\int_0^T f(t+\varepsilon)\,d\varepsilon = \frac{1}{T}\int_t^{T+t} f(\eta)\,d\eta = \frac{1}{T}\int_0^T f(\eta)\,d\eta = const$$

$$R_X(t,s) = E[X(t)X(s)] = \frac{1}{T}\int_0^T f(t+\varepsilon)f(s+\varepsilon)\,d\varepsilon$$

$$= \frac{1}{T}\int_t^{T+t} f(\eta)f(\eta+s-t)\,d\eta$$

$\because f(\eta+T)f(\eta+s-t+T) = f(\eta)f(\eta+s-t) \triangleq f(\eta)f(\eta-\tau)$

$\therefore R_X(t,s) = \frac{1}{T}\int_0^T f(\eta)f(\eta-\tau)\,d\eta = R_X(\tau)$.

$\therefore$ 宽平稳.

2. 解:

$$E[X(t)] = E(A)\,E[\cos(\omega t+\theta)] = E(A)\cdot 0 = 0.$$

$$R_X(t,s) = E[A^2\cos(\omega t+\theta)\cos(\omega s+\theta)] = E\left[\frac{A^2}{2}\{\cos[\omega(t+s)+2\theta] + \cos[\omega(t-s)]\}\right]$$

$$= \frac{1}{2}E[A^2]\cos\omega(t-s)$$

$$= \frac{1}{2}E[A^2]\cos\omega\tau = R_X(\tau)$$

$\therefore$ 宽平稳.

3. 解: $\because R_X(\tau)$ 连续, $R_Y(\tau)$ 连续

$\therefore X(t)$、$Y(t)$ 皆均方连续.

$\because R_X'(\tau) = -2\alpha\tau e^{-\alpha\tau^2}$, $R_X''(\tau) = -2\alpha e^{-\alpha\tau^2} + 4\alpha^2\tau^2 e^{-\alpha\tau^2}$ 连续,

$\therefore X(t)$ 均方可导.

$\because R_Y'(\tau)$ 不连续. $\therefore Y(t)$ 非均方可导.

4. 解:

$$\cancel{R_Y(\tau) = E\left\{\left[X(t+\tau)+X'(t+\tau)\right]\left[\overline{X(t)+X'(t)}\right]\right\}}$$
$$\cancel{= E\left[X(t+\tau)\overline{X(t)}\right] + E\left[X(t+\tau)\overline{X'(t)}\right]}$$
$$\cancel{+ E\left[X'(t+\tau)\overline{X(t)}\right] + E\left[X'(t+\tau)\overline{X'(t)}\right]}$$
$$\cancel{= R_X(\tau) + E\left\{\left[X(t+\tau)\overline{X(t)}\right]'\right\}} + R_{X'}(\tau)$$

$$R_Y(t,s) = E\left\{\left[X(t)+X'(t)\right]\left[\overline{X(s)+X'(s)}\right]\right\}$$
$$= E\left[X(t)\overline{X(s)}\right] + E\left[X(t)\overline{X'(s)}\right] + E\left[X'(t)\overline{X(s)}\right] + E\left[X'(t)\overline{X'(s)}\right]$$
$$= R_X(t-s) + \frac{\partial}{\partial s}R_X(t-s) + \frac{\partial}{\partial t}R_X(t-s) + \frac{\partial^2}{\partial t\partial s}R_X(t-s)$$
$$= R_X(\tau) - \frac{\partial}{\partial \tau}R_X(\tau) + \frac{\partial}{\partial \tau}R_X(\tau) - \frac{\partial^2}{\partial \tau^2}R_X(\tau)$$
$$= e^{-\tau^2} - \left(-2e^{-\tau^2} + 4\tau^2 e^{-\tau^2}\right)$$
$$= 3e^{-\tau^2} - 4\tau^2 e^{-\tau^2}.$$

5. 解:

$$E[Y(t)] = \frac{1}{t}\int_0^t E[X(s)]\,ds = \frac{1}{t}\int_0^t \cos 3s\,ds = Sa(3t).$$

$$\cancel{R_Y(t,s) = E\left[\frac{1}{t}\int_0^t X(u)du\ \frac{1}{s}\int_0^s X(v)dv\right]}$$
$$\cancel{= \frac{1}{st}\int_0^t\int_0^s E(\xi^2)\cos 3u\cos 3v\,du\,dv}$$
$$\cancel{= \frac{1}{9st}Sa(3t)Sa(3s)}$$

$$C_Y(t,s) = E\left\{\left[\frac{1}{t}\int_0^t X(u)du - Sa(3t)\right]\left[\frac{1}{s}\int_0^s X(v)dv - Sa(3s)\right]\right\}$$
$$= E\left[\frac{1}{t}\int_0^t X(u)du\cdot\frac{1}{s}\int_0^s X(v)dv\right] - Sa(3t)Sa(3s).$$
$$= \frac{1}{st}\int_0^t\int_0^s E[X(u)X(v)]\,du\,dv - Sa(3t)Sa(3s)$$
$$= \frac{1}{st}\int_0^t\int_0^s E(\xi^2)\cos 3u\cos 3v\,du\,dv - Sa(3t)Sa(3s).$$
$$= \frac{2}{st}\int_0^t\int_0^s \cos 3u\cos 3v\,du\,dv - Sa(3t)Sa(3s)$$

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清華大學

谱分析部分

1. 解：

$$Z(t) \triangleq Y(t) - X(t)$$

$$R_Z(t,s) = E[Z(t)\overline{Z(s)}]$$

$$= E\{[Y(t)-X(t)][\overline{Y(s)-X(s)}]\}$$

$$= E[Y(t)\overline{Y(s)}] - E[Y(t)\overline{X(s)}] - E[X(t)\overline{Y(s)}] + E[X(t)\overline{X(s)}]$$

$$= R_Y(t-s) - R_{YX}(t-s) - R_{XY}(t-s) + R_X(t-s)$$

$$= R_X(t-s) * h(\tau) * \overline{h(-\tau)} - R_X(\tau) * h(\tau) - R_X(\tau) * \overline{h(-\tau)} + R_X(\tau)$$

$$\therefore S_Z(\omega) = |H(\omega)|^2 S_X(\omega) - S_X(\omega) H(\omega) - S_X(\omega)\overline{H(\omega)} + S_X(\omega).$$

$$= [|H(\omega)|^2 - H(\omega) - \overline{H(\omega)} + 1] S_X(\omega).$$

$$= (\overline{H}(\omega) - 1)(H(\omega) - 1) S_X(\omega).$$

$$= |H(\omega) - 1|^2 S_X(\omega).$$

2. 解：(1) $Y(t) = X(t) - X(t-T)$

$$\therefore Y(s) = X(s) - X(s)e^{-sT} = (1-e^{-sT})X(s).$$

$$Z(t) = \int_{-\infty}^{t} Y(\lambda)\,d\lambda.$$

$$\therefore Z(s) = \frac{Y(s)}{s}.$$

$$\therefore Z(s) = \frac{1-e^{-sT}}{s} X(s)$$

(2) $H(j\omega) = \dfrac{1-e^{-j\omega T}}{j\omega} = \dfrac{1-\cos\omega T + j\sin\omega T}{j\omega}.$

$$\therefore |H(j\omega)|^2 = \frac{(1-\cos\omega T)^2 + \sin^2\omega T}{\omega^2} = \frac{2(1-\cos\omega T)}{\omega^2} \quad \left(= \frac{4\sin^2\frac{\omega T}{2}}{\omega^2}\right).$$

又 $R_X(\tau) = S_0\delta(\tau)$，$\therefore S_X(\omega) = S_0.$

$$\therefore S_Z(\omega) = 2S_0(1-\cos\omega T)/\omega^2 \quad \therefore D(Z(t)) = R_Z(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Z(\omega)\,d\omega = S_0 T.$$

3. 解：

$$|H(\omega)|^2 = \frac{\omega^2+8}{\omega^2+3}$$

$$\cancel{|H(s)|^2 = \frac{-s^2+8}{-s^2+3} = \frac{s^2-8}{s^2-3} = \frac{(s+\sqrt{8})(s-\sqrt{8})}{(s+\sqrt{3})(s-\sqrt{3})}}$$

$$H(j\omega)H(-j\omega) = \frac{(j\omega+\sqrt{8})(-j\omega+\sqrt{8})}{(j\omega+\sqrt{3})(-j\omega+\sqrt{3})}.$$

若求因果、稳定，则 $H(j\omega) = \frac{j\omega+\sqrt{8}}{j\omega+\sqrt{3}}$ 或 $\frac{-j\omega+\sqrt{8}}{j\omega+\sqrt{3}}$.

若要求最小相位系统，则 $H(j\omega) = \frac{j\omega+\sqrt{8}}{j\omega+\sqrt{3}}$.

4. 解：设 $H_1(f)$ 输出信号为 $Z(t)$，乘载波后信号为 $W(t)$.

$$W(t) = Z(t)\cos(\omega_c t+\theta).$$

$$\begin{aligned}
R_W(t,s) &= E[W(t)W(s)] \\
&= E\left[Z(t)\cos(\omega_c t+\theta)Z(s)\cos(\omega_c s+\theta)\right] \\
&= E[Z(t)Z(s)]\,E\left[\cos(\omega_c t+\theta)\cos(\omega_c s+\theta)\right] \\
&= R_Z(t-s)\cdot\frac{1}{2}E\left\{\cos(\omega_c(t+s)+2\theta)+\cos[\omega_c(t-s)]\right\} \\
&\doteq \frac{1}{2}R_Z(\tau)\cdot\cos\omega_c(t-s)
\end{aligned}$$

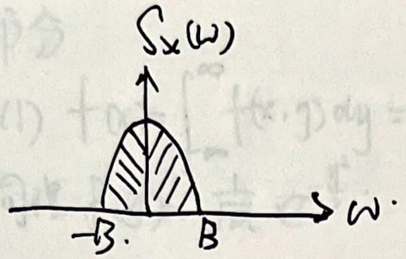
$$\therefore S_W(\omega) = \frac{1}{4}\left[S_Z(\omega+\omega_c)+S_Z(\omega-\omega_c)\right]\cdot\frac{N_0}{2}$$

经 $H_2(f)$ 后，$S_Y(\omega) = \frac{1}{2}\left[u(\omega+2\pi B)-u(\omega-2\pi B)\right]\cdot\frac{N_0}{2}$

$\because E[W(t)]=0 \quad \therefore E[Y(t)]=0$

$$\therefore D[Y(t)] = R_Y(0) = \frac{1}{2\pi}\int S_Y(\omega)\,d\omega = B\cdot\frac{N_0}{2}.$$

5. 解：



$$R_Y(\tau)=E\left[\left(X(t)\cos\omega_0 t-\hat{X}(t)\sin\omega_0 t\right)\overline{\left(X(s)\cos\omega_0 s-\hat{X}(s)\sin\omega_0 s\right)}\right]$$

$$=E\left(X(t)\overline{X(s)}\cos\omega_0 t\cos\omega_0 s\right)-E\left(X(t)\overline{\hat{X}(s)}\cos\omega_0 t\sin\omega_0 s\right)$$
$$-E\left(\hat{X}(t)\overline{X(s)}\sin\omega_0 t\cos\omega_0 s\right)+E\left(\hat{X}(t)\overline{\hat{X}(s)}\sin\omega_0 t\sin\omega_0 s\right)$$

$$=R_X(\tau)\cos\omega_0 t\cos\omega_0 s-R_{X\hat{X}}(\tau)\cos\omega_0 t\sin\omega_0 s$$
$$-R_{\hat{X}X}(\tau)\sin\omega_0 t\cos\omega_0 s+R_{\hat{X}}(\tau)\sin\omega_0 t\sin\omega_0 s$$

$$=R_X(\tau)\cos\omega_0 t\cos\omega_0 s+R_{\hat{X}X}(\tau)\cos\omega_0 t\sin\omega_0 s$$
$$-R_{\hat{X}X}(\tau)\sin\omega_0 t\cos\omega_0 s+R_X(\tau)\sin\omega_0 t\sin\omega_0 s.$$

$$=R_X(\tau)\cos\omega_0\tau-R_{\hat{X}X}(\tau)\sin\omega_0\tau.$$

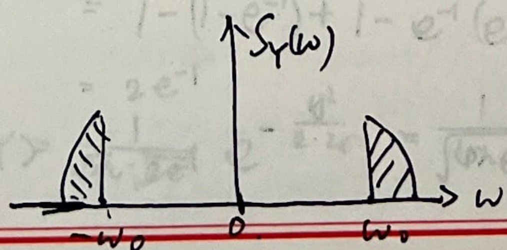
$$\therefore S_Y(\omega)=\frac{1}{2}\left[S_X(\omega+\omega_0)+S_X(\omega-\omega_0)\right]-\frac{1}{2j}\left[S_{\hat{X}X}(\omega-\omega_0)-S_{\hat{X}X}(\omega+\omega_0)\right]$$

$$=\frac{1}{2}\left[S_X(\omega+\omega_0)+S_X(\omega-\omega_0)\right]-\frac{1}{2j}\left\{S_X(\omega-\omega_0)\left[-j\,\mathrm{sgn}(\omega-\omega_0)\right]-S_X(\omega+\omega_0)\left[-j\,\mathrm{sgn}(\omega+\omega_0)\right]\right\}$$

$$=\frac{1}{2}\left[S_X(\omega+\omega_0)+S_X(\omega-\omega_0)\right]+\frac{1}{2}\left[S_X(\omega-\omega_0)\,\mathrm{sgn}(\omega-\omega_0)-S_X(\omega+\omega_0)\,\mathrm{sgn}(\omega+\omega_0)\right]$$

$$=\frac{1}{2}S_X(\omega-\omega_0)\left[1+\mathrm{sgn}(\omega-\omega_0)\right]+\frac{1}{2}S_X(\omega+\omega_0)\left[1-\mathrm{sgn}(\omega+\omega_0)\right].$$

$$=S_X(\omega-\omega_0)\,u(\omega-\omega_0)+S_X(\omega+\omega_0)\,u(-\omega-\omega_0).$$



高斯过程部分

1. 证明: (1) $f(x)=\int_{-\infty}^{\infty} f(x,y)\,dy = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

同理 $f(y)=\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$.

(2) $E(X)=0$, $E(Y)=0$

$$\therefore C_{XY}=R_{XY}=E(XY)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy$$

$$=\int_{-\infty}^{\infty}\frac{x}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx\int_{-\infty}^{\infty}\frac{y}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy+\frac{1}{2\pi}e^{-\pi^2}\int_{-\infty}^{\infty}xg(x)\,dx\int_{-\infty}^{\infty}yg(y)\,dy$$

$=0$.

但 $f(x,y)\neq f(x)\cdot f(y)$

2. 解: Y 是 $X(t)$ 过线性系统输出的信号, 故 $Y\sim N$

$\because R_X(t)$ 连续

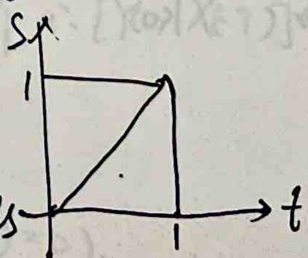
$\therefore X$ 均方可积

$$\therefore E(Y^2)=E\left[\int_0^1 X(t)\,dt\int_0^1 X(s)\,ds\right]$$

$$=\int_0^1\int_0^1 E[X(t)X(s)]\,dt\,ds.$$

$$=\int_0^1\int_0^1 e^{-|t-s|}\,dt\,ds.$$

$$=\int_0^1\int_0^s e^{-(s-t)}\,dt\,ds+\int_0^1\int_s^1 e^{-(t-s)}\,dt\,ds$$



$$=\int_0^1 e^{-s}(e^s-1)\,ds+\int_0^1 e^s(e^{-s}-e^{-1})\,ds$$

$$=1-(1-e^{-1})+1-e^{-1}(e^1-1)$$

$$=2e^{-1}$$

$$\therefore f(Y)=\frac{1}{\sqrt{2\pi}\cdot\sqrt{2e^{-1}}}e^{-\frac{y^2}{2\cdot 2e^{-1}}}=\frac{1}{\sqrt{4\pi e^{-1}}}e^{-\frac{y^2}{4e^{-1}}}.$$

3. 解：(1) $Y(t)$ 也是高斯过程.

$$E[Y(t)] = 0.$$

$$\cancel{R_Y(t,s) = \sigma^2 \int_0^{\infty}\int_0^{\infty} e^{-\alpha|t-s|}\,dt\,ds}$$

$$H(j\omega) = \frac{1}{j\omega+\beta}. \quad |H(j\omega)|^2 = \frac{1}{\omega^2+\beta^2}.$$

$$S_X(\omega) = \frac{\sigma^2}{\alpha}\frac{2}{(\omega/\alpha)^2+1}.$$

$$\therefore S_Y(\omega) = \frac{2\sigma^2}{\alpha}\frac{1}{(\omega/\alpha)^2+1}\cdot\frac{1}{\omega^2+\beta^2}$$

$$= \frac{2\sigma^2}{\alpha}\left[\frac{1/(\beta^2-\alpha^2)}{(\omega/\alpha)^2+1} - \frac{\alpha^2/(\beta^2-\alpha^2)}{\omega^2+\beta^2}\right] = \frac{2\sigma^2}{\alpha(\beta^2-\alpha^2)}\left[\frac{1}{(\omega/\alpha)^2+1} - \frac{\alpha^2}{\beta^2}\cdot\frac{1}{(\omega/\beta)^2+1}\right]$$

$$\therefore R_Y(\tau) = \frac{\sigma^2}{\alpha(\beta^2-\alpha^2)}\cdot\left[\alpha e^{-\alpha|\tau|} - \frac{\alpha^2}{\beta^2}\beta e^{-\beta|\tau|}\right]$$

$$\therefore R_Y(0) = \frac{\sigma^2}{\beta^2-\alpha^2}\left(1-\frac{\alpha}{\beta}\right) = \frac{\sigma^2}{\beta(\beta+\alpha)}.$$

$$\therefore f_Y(y) = \frac{\sqrt{\beta(\beta+\alpha)}}{\sqrt{2\pi}\,\sigma}\exp\left\{-\frac{y^2\beta(\beta+\alpha)}{2\sigma^2}\right\}.$$

$$\therefore P\{Y(0)\geq r\} = Q\left(\frac{r\beta(\beta+\alpha)}{\sigma}\right).$$

(2). $\because [X, Y]$ 联合高斯 $\therefore [X(-T), Y(0)]$ 联合高斯 $\therefore [Y(0)|X(-T)]$ 条件高斯.

$$E[Y(0)] = E\left[\int_{-\infty}^{\infty} h(\tau)x(-\tau)\,d\tau\right] = 0.$$

$$E[X(-T)] = 0$$

$$\therefore E[Y(0)|X(-T)] = \frac{C_{Y(0)X(-T)}}{C_{X(-T)X(-T)}}[X(-T)] = 0. \qquad (X(-T)=0).$$

$$C_{Y(0)|X(-T)} = C_{Y(0)Y(0)} - C_{Y(0)X(-T)}C_{X(-T)Y(0)}/C_{X(-T)X(-T)}$$

$$= R_{Y(0)} - R_{Y(0)X(-T)}|^2/R_{X(-T)}$$

$$\because R_{Y(0)X(-T)} = E[Y(0)X(-T)] = E\left[\int_{-\infty}^{\infty} h(\tau)x(-\tau)X(-T)\,d\tau\right]$$

$$= \sigma^2\int_{-\infty}^{\infty} h(\tau)\,e^{-\alpha|\tau-T|}\,d\tau$$

$$= \sigma^2 \int_{-\infty}^{\infty} u(\tau) e^{-\beta\tau - \alpha|\tau - T|} d\tau$$

$$= \sigma^2 \int_0^T e^{-\beta\tau - \alpha(T-\tau)} d\tau + \sigma^2 \int_T^\infty e^{-\beta\tau - \alpha(\tau - T)} d\tau$$

$$= \sigma^2 e^{-\alpha T} \int_0^T e^{(\beta+\alpha)\tau} d\tau + \sigma^2 e^{\alpha T} \int_T^\infty e^{-(\beta+\alpha)\tau} d\tau$$

$$= \frac{\sigma^2 e^{-\alpha T}}{\alpha - \beta}\left[e^{(\alpha-\beta)T} - 1\right] + \frac{\sigma^2 e^{\alpha T}}{\alpha + \beta} e^{-(\beta+\alpha)T}$$

$$= \frac{\sigma^2}{\alpha - \beta} e^{-\beta T} - \frac{\sigma^2 e^{-\alpha T}}{\alpha - \beta} + \frac{\sigma^2}{\alpha + \beta} e^{-\beta T}.$$

$$\therefore C_{Y(0)|X(-T)} = \frac{\sigma^2}{\beta(\beta+\alpha)} - \left|\frac{\sigma^2(e^{-\beta T} - e^{-\alpha T})}{\alpha - \beta} + \frac{\sigma^2}{\alpha+\beta} e^{-\beta T}\right|^2 / \sigma^2.$$

$$= \sigma^2\left[\frac{1}{\beta(\beta+\alpha)} - \left(\frac{e^{-\beta T} - e^{-\alpha T}}{\alpha - \beta} + \frac{e^{-\beta T}}{\alpha+\beta}\right)^2\right] \triangleq \sigma^2 A$$

$$\therefore f_{Y(0)|X(-T)=0}(y) = \frac{A^{-\frac{1}{2}}}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{y^2}{2\sigma^2 A}\right\}.$$

$$P\{Y(0) \geq 0 \mid X(-T) = 0\} = Q\left\{\frac{\gamma}{\sigma\sqrt{A}}\right\}.$$

5. 解: $Y(t) = Z\cos\omega t\cos\theta - Z\sin\omega t\sin\theta.$

作变换 $\begin{pmatrix} Z\cos\theta \\ -Z\sin\theta \end{pmatrix} \Rightarrow \begin{pmatrix} u \\ v \end{pmatrix}$ $\therefore$ $Z = \sqrt{u^2+v^2}$ $(Z \geq 0)$.

$\theta = -\mathrm{arctg}\frac{v}{u}.\ \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right).$

$$\therefore f(u,v) = f(z,\theta)\cdot\left|\frac{\partial(z,\theta)}{\partial(u,v)}\right| = f(z)f(\theta)\det\begin{vmatrix} \frac{u}{\sqrt{u^2+v^2}} & \frac{v}{\sqrt{u^2+v^2}} \\ \frac{v}{u^2+v^2} & -\frac{u}{u^2+v^2} \end{vmatrix}$$

$$= \frac{z}{\sigma^2}\exp\left\{\frac{z^2}{2\sigma^2}\right\}\cdot\frac{1}{2\pi}\cdot\frac{1}{\sqrt{u^2+v^2}}$$

$$= \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{u^2+v^2}{2\sigma^2}\right\} \sim N\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\sigma^2 & 0 \\ 0 & \sigma^2\end{pmatrix}\right\}.$$

$$\therefore f_{Y(t)}(y) = N\left\{(\cos\omega t, \sin\omega t)\begin{pmatrix}0\\0\end{pmatrix}, (\cos(\omega t), \sin(\omega t))\begin{pmatrix}\sigma^2 & 0 \\ 0 & \sigma^2\end{pmatrix}\begin{pmatrix}\cos\omega t \\ \sin\omega t\end{pmatrix}\right\}$$

$$= N(0, \sigma^2).$$

$Y(t)$ 和 ~~$f_{X(t)}(x)$~~ $X(t)$ 的一维分布相同 $\therefore Y(t)$ 和 $X(t)$ 的有限维分布相同（高斯才满足）

泊松过程部分

1. 解：(1). $\Phi_{X_1}(j\omega)=\exp\{\lambda_1 t(e^{j\omega}-1)\}$, $\Phi_{X_2}(j\omega)=\exp[\lambda_2 t(e^{j\omega}-1)]$

∵ $X_1(t)$ $X_2(t)$ 独立

∴ $\Phi_{X_1(t)+X_2(t)}(j\omega)=\exp[(\lambda_1+\lambda_2)t(e^{j\omega}-1)]$

∴ $X_1(t)+X_2(t)\sim P(\lambda_1+\lambda_2)$.

(2). $\Phi_{X_1(t)-X_2(t)}(j\omega)=E[e^{j\omega[X_1(t)-X_2(t)]}]$

$=E\{e^{j\omega X_1(t)}\}\cdot E\{e^{j(-\omega)X_2(t)}\}$

$=\exp\{\lambda_1 t(e^{j\omega}-1)\}\cdot\exp\{\lambda_2 t(e^{-j\omega}-1)\}$

$=\exp\{(\lambda_1+\lambda_2)t(\frac{\lambda_1}{\lambda_1+\lambda_2}e^{j\omega}+\frac{\lambda_2}{\lambda_1+\lambda_2}e^{-j\omega}-1)\}$

∴ $X_1(t)-X_2(t)$ 非泊松，是复合了两点分布

X	1	-1
$P(x)$	$\frac{\lambda_1}{\lambda_1+\lambda_2}$	$\frac{\lambda_2}{\lambda_1+\lambda_2}$

为复合泊松，强度 $\lambda_1+\lambda_2$.

2. 解: $E\{N_{t_0}(t)\}=\int_{t_0}^{t_0+t}\lambda(\tau)d\tau=0.5\int_{t_0}^{t_0+t}(1+\cos\omega\tau)d\tau=0.5\{t+\frac{1}{\omega}[\sin\omega(t_0+t)-\sin\omega t_0$

$D\{N_{t_0}(t)\}=E\{N_{t_0}(t)\}$.

3. 解: 构造随机变量 Z_k:

Z	1	0
$P(Z_k)$	p	$1-p$

∴ $Y(t)=\sum_{k=1}^{X(t)}Z_k$

∴ $Y(t)$ 为一复合泊松过程. , $\Phi_{Y(t)}(j\omega)=\exp\{\lambda t\cdot[\Phi_Z(j\omega)-1]\}$

$$= \exp\{\lambda t[p e^{j\omega\cdot 1}+(1-p)e^{j\omega\cdot 0}-1]\}$$

$$= \exp\{\lambda t[p e^{j\omega}-p]\}$$

$$= \exp\{\lambda p t[e^{j\omega}-1]\}$$

$\therefore Y(t)$ 为一强度为 λp 的泊松过程.

解： $\because X(t)$ 只能取完全平方数

$\therefore X(t)$ 非泊松过程，一维分布不服从泊松分布.

$$E\{X(t)\} = E[N^2(t)]$$

$$= \sum_{n=0}^{\infty} n^2 \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

$$= e^{-\lambda t}\sum_{n=1}^{\infty} n\lambda t \frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$= e^{-\lambda t}\lambda t\sum_{n=1}^{\infty}(n-1+1)\frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$= \lambda t e^{-\lambda t}\left[\sum_{n=2}^{\infty}\frac{(\lambda t)^{n-2}}{(n-2)!}\lambda t + \sum_{n=1}^{\infty}\frac{(\lambda t)^{n-1}}{(n-1)!}\right]$$

$$= \lambda t\, e^{-\lambda t}\cdot e^{\lambda t}(\lambda t+1).$$

$$= (\lambda t)^2+\lambda t$$

$$E[X^2(t)] = E[N^4(t)]$$

$$= \sum_{n=0}^{\infty} n^4\frac{(\lambda t)^n}{n!}e^{-\lambda t}$$

$$= e^{-\lambda t}\lambda t\sum_{n=1}^{\infty} n^3\frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$= \lambda t e^{-\lambda t}\sum_{n=1}^{\infty}\{(n-1)^3+3(n-1)^2+3(n-1)+1\}\frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$= \lambda t e^{-\lambda t}\left\{\sum_{n=2}^{\infty}(n-1)^2\frac{(\lambda t)^{n-2}}{(n-2)!}\lambda t + \sum_{n=2}^{\infty}3(n-1)\cdot\lambda t\frac{(\lambda t)^{n-2}}{(n-2)!} + \sum_{n=2}^{\infty}3\lambda t\frac{(\lambda t)^{n-2}}{(n-2)!} + \sum_{n=1}^{\infty}\frac{(\lambda t)^{n-1}}{(n-1)!}\right\}$$

$$= \lambda t e^{-\lambda t}\left\{\sum_{n=2}[(n-2)^2+2(n-2)+1]\frac{(\lambda t)^{n-2}}{(n-2)!}\lambda t + \sum_{n=2}3\lambda t[(n-2)+1]\frac{(\lambda t)^{n-2}}{(n-2)!} + (3\lambda t+1)e^{\lambda t}\right.$$

$$= \lambda t e^{-\lambda t} \left\{ \sum_{n=2}^{\infty} \lambda t (n-2)^2 \frac{(\lambda t)^{n-2}}{(n-2)!} + \sum_{n=2}^{\infty} 5\lambda t (n-2) \frac{(\lambda t)^{n-2}}{(n-2)!} + 7\lambda t e^{\lambda t} + e^{\lambda t} \right\}$$

$$= \lambda t e^{-\lambda t} \left\{ [(\lambda t)^2 + (\lambda t)^2] e^{\lambda t} + 5(\lambda t)^2 e^{\lambda t} + 7\lambda t e^{\lambda t} + e^{\lambda t} \right\}$$

$$= (\lambda t)^4 + 6(\lambda t)^3 + 7(\lambda t)^2 + \lambda t. \qquad ?$$

5. 解：

$$E[M(t)] = \frac{1}{T}\int_0^T E[X(t)]\,dt = \frac{\lambda}{T}\int_0^T t\,dt = \frac{\lambda}{2T}T^2 = \frac{1}{2}\lambda T.$$

$$D[M(t)] = E\left\{\left[\frac{1}{T}\int_0^T X(t)\,dt - \frac{1}{2}\lambda T\right]^2\right\}$$

$$= E\left\{\left(\frac{1}{T}\int_0^T X(t)\,dt\right)^2\right\} - \left(\frac{1}{2}\lambda T\right)^2$$

$$= \frac{1}{T^2}\int_0^T\int_0^T E\{X(t)X(s)\}\,dt\,ds - \left(\frac{1}{2}\lambda T\right)^2$$

$$\because \Phi_{X(t)}(j\omega) = \exp\{\lambda t(e^{j\omega}-1)\} = E\{e^{j\omega X}\}.$$

$$\therefore \Phi'_{X(t)}(j\omega) = \lambda t j e^{j\omega}\exp\{\lambda t(e^{j\omega}-1)\} = E\{jX e^{j\omega X}\}.$$

$$\therefore \Phi''_{X(t)}(j\omega) = \lambda t (j)^2 e^{j\omega}\exp\{\lambda t(e^{j\omega}-1)\} + (\lambda t)^2 j^2 e^{2j\omega}\exp\{\lambda t(e^{j\omega}-1)\}$$

$$= E\{(jX)^2 e^{j\omega X}\}$$

$$\therefore E[X^2] = \frac{1}{(j)^2}\cdot\Phi''_{X(t)}(j\omega)\Big|_{\omega=0} = \frac{1}{(j)^2}[\lambda t (j)^2 + (\lambda t)^2 (j)^2] = \lambda t + (\lambda t)^2$$

(上方法得不出 $R_X(t,s)$)，由于泊松过程的自相关函数

$$R_X(t,s) = \lambda^2 st + \lambda\min\{s,t\},$$

$$\therefore D[M(t)] = \frac{1}{T^2}\int_0^T\int_0^T \lambda^2 st\,dt\,ds + \frac{1}{T^2}\int_0^T\int_0^T \lambda\min\{s,t\}\,dt\,ds - \left(\frac{1}{2}\lambda T\right)^2.$$

$$= \frac{\lambda^2}{T^2}\left(\frac{1}{2}T^2\right)^2 + \frac{1}{T^2}\int_0^T\int_0^s \lambda t\,dt\,ds + \frac{1}{T^2}\int_0^T\int_s^T \lambda s\,dt\,ds - \left(\frac{1}{2}\lambda T\right)^2.$$

$$= \frac{\lambda}{T^2}\int_0^T \frac{1}{2}s^2\,ds + \frac{\lambda}{T^2}\int_0^T s(T-s)\,ds$$

$$= \frac{\lambda}{T^2}\frac{1}{6}T^3 + \frac{\lambda}{T^2}\frac{1}{6}T^3$$

$$= \frac{1}{3}\lambda T.$$